Solar Influence on Satellite Motion Near the Stable Earth-Moon Libration Points

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The motion of a space vehicle in the vicinity of the stable earth-moon libration points has been studied using a model which includes the perturbing effects of the gravitational attraction and the radiation pressure of the sun The earth and the moon are assumed to move in circular orbits about their mass center, and the earth-moon orbit plane is assumed to be inclined at an angle of 5°9′ to the ecliptic The differential equations of motion in a libration point centered coordinate system were integrated numerically to determine the behavior of the vehicle The results of the investigation indicate that, when the effects of the sun are included, the natural motions of a vehicle placed initially at L_4 or L_5 will not remain near the respective point Although the vehicle does move on a trajectory about the libration point for at least 700 days, the influence of the sun causes the vehicle to move through wide departures from the libration point The effects of the solar-radiation pressure for several satellite area-to-mass ratios are studied and the impulse requirements necessary to force a vehicle to remain at L_4 or at a point near L_4 are deteı mined

Nomenclature

A = projected area of spacecraft on a plane perpendicular to the spacecraft-sun line
d = distance between the earth center and the moon center
G = universal gravitational constant
i = inclination of the earth-moon plane to the ecliptic plane
k = solar-pressure constant
m = mass of the body denoted by the accompanying subscript
R = position vector of the eaı th-moon barycenter relative to the center of the sun
r = position vector from the body denoted by the first subscript to the body denoted by the second subscript
S = magnitude of the force due to the solar-radiation pressure
U = gravitational force potential of sun
V = potential function defined in Eq (5)
X, Y, Z = displacement values in the barycenter centered inertial coordinate system
x, y, z = displacement values in the libration point, centered-rotating coordinate system
δ = declination of the sun relative to the barycenter, centered inertial coordinate system
θ = angle between the X axis and the earth-moon line measured relative to the barycenter-interial coordinate system
ρ = position vector from the earth-moon barycenter to the spacecraft
Φ = angle between the X axis and the projection of the radius vector to the sun on the earth-moon plane, and measured in the barycenter-inertial coordinate system
ψ = angle between the X axis and the radius vector to the sun measured in the ecliptic plane
Ω = angular velocity of the earth-moon barycenter about the sun
ω = angular velocity of the earth-moon line about the earth moon barycenter
ξ, η, ζ = displacement values in the barycenter, centered rotating coordinate system

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Introduction

IN the space that surrounds two bodies that orbit about their mutual mass center, there are five points at which a third body will remain in equilibrium under the gravitational attractions of the other two bodies The existence of these points was predicted by Lagrange in 1772 As shown in Fig 1, three of the points lie on a straight line that joins the mass centers of the first two bodies The two remaining points lie at the vertices of two equilateral triangles which have the line joining the two bodies as a mutual base In establishing the existence of the five libration points, the gravitational attractions of other celestial bodies are neglected Since, in any realizable physical system, other bodies will be present, the nature of the motion when the third body suffers a small disturbance is of interest Discussions of this problem are given in Refs 1 and 2

More recently, the motion of a particle in the vicinity of the libration points associated with the earth-moon configuration has been considered in a number of studies,[3–8] and several possible reasons for attempting to establish an artificial satellite at these points are discussed However, each of the investigations neglects the effects of the sun and the other planets

In Ref 9, the influence of the sun on the stability of the motion near an earth-moon libration point is discussed Lyapunov's criterion is used to demonstrate that there are points in the very near region of L_4 and L_5 where the motion will not be stable However, the study assumes that the orbits of the earth, the moon, and the sun are coplanar, and it neglects the influence of the sun on the earth and the moon as well as the motion of the earth-moon mass center about the sun Furthermore, a description of the motion is not given and the question of whether or not the particle actually escapes from a trajectory around the libration point is not answered That is, although the motion may be unstable very near the libration point, stable motion may be possible in another region around the libration point In the subsequent discussion the term "stable" is used to imply that, if the motion is stable, the particle will remain within a certain region for the period of time during which the motion is studied The existence of such regional stability is suggested by the reported sightings of two cloud-like objects in the vicinity of the trailing earth-moon libration point L_5 [10 11]

Following the report of these objects, an investigation was started to determine 1) whether or not a space vehicle placed at either L_4 or L_5 would remain for an indefinite period in the vicinity of the L_4 and L_5 points and 2) what the nature of the motion would be for periods of a few years The effects of both the solar-gravitational attraction and the solar-radiation pressure are included in the analysis The earth-moon orbit plane is assumed to be inclined to the ecliptic at an angle of 5°9′ However, the eccentricity of the moon's orbit about the earth, the effects of the nonsphericity of the earth and the moon, and the effects of the other planets are neglected

Equations of Motion

The coordinate system used to describe the motion is shown in Fig 2 The earth and the moon move in circular orbits about their mass center, and the mass center, in turn, moves in a circular orbit about the sun The earth-moon orbit plane is inclined at an angle of 5°9′ to the ecliptic Each of the bodies is treated as a mass point

The X,Y,Z coordinate system is centered at the earth-moon mass center and selected so that the X axis points towards the vernal equinox of date The Y axis lies in the earth-moon orbit plane and is chosen so that the Z axis points in the direction of the angular velocity vector for the earth-moon configuration The ξ,η,ζ coordinate system is centered at the earth-moon mass center and selected so that the ξ axis always lies along the earth-moon line, the η axis lies in the earth-moon orbit plane, and the ζ axis coincides with the Z axis The ξ, η coordinates rotate about the ζ axis with the same angular velocity ω as the earth-moon line In the rotating ξ,η,ζ coordinate system, the differential equations governing the motion of a vehicle of infinitesimal mass are

$$\begin{aligned} \ddot{\xi} - 2\omega\dot{\eta} - \omega^2\xi &= -\frac{\partial V}{\partial \xi} + \frac{\partial U}{\partial \xi} + S_\xi \\ \ddot{\eta} + 2\omega\dot{\xi} - \omega^2\eta &= -\frac{\partial V}{\partial \eta} + \frac{\partial U}{\partial \eta} + S\eta \\ \ddot{\zeta} &= -\frac{\partial V}{\partial \zeta} + \frac{\partial U}{\partial \zeta} + S_\zeta \end{aligned} \tag{1}$$

where ( ) indicates differentiation with respect to time V and U are given by

$$\begin{aligned} V &= \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2} \\ U &= Gm_3\left[\frac{1}{\rho_3} - \frac{\xi\xi_3 + \eta\eta_3 + \zeta\zeta_3}{R^3}\right] \end{aligned} \tag{2}$$

where G is the universal gravitation constant, and m_1, m_2, and m_3 are the mass of the earth, the moon, and the sun, respectively ρ_1, ρ_2, and ρ_3 are the magnitudes of the distance of the earth, the moon, and the sun, respectively, from the vehicle, and R is the magnitude of the position vector of the sun in the ξ,η,ζ coordinates S_ξ, S_η, and S_ζ are the respective components of the solar-radiation pressure

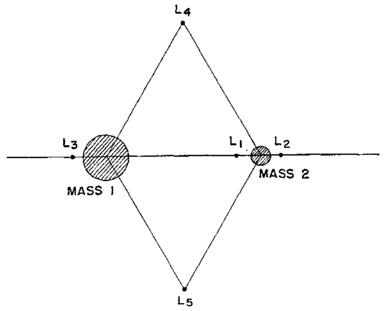

Fig 1 Libration points in the restricted three-bod problem

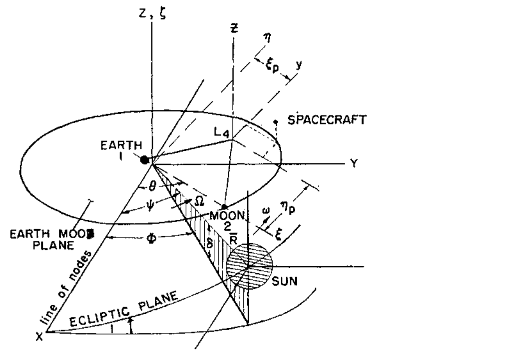

Fig 2 Three-dimensional illustration of the coordinate systems and spacecraft location geometry

Equations (1) can be transferred to the libration point centered coordinate system x,y,z, by the following transformations:

$$\begin{aligned} \xi &= x + \xi_p \\ \eta &= y + \eta_p \\ \zeta &= z \end{aligned} \tag{3}$$

where ξ_p and η_p are coordinates of the libration point and are constant If the appropriate derivatives of Eqs (2) and (3) are substituted, along with Eqs (3), into Eqs (1), the differential equations of motion in the x,y,z coordinate system become

$$\left.\begin{aligned} \ddot{x} &= 2\omega\dot{y} + (x + \xi_p)\omega^2 - (x_3 + \xi_p)\Omega^2 + S_x + \sum_{i=1}^{3} \frac{Gm_i}{\rho_i^3}(x_i - x) \\ \ddot{y} &= -2\omega\dot{x} + (y + \eta_p)\omega^2 - (y_3 + \eta_p)\Omega^2 + S_y + \sum_{i=1}^{3} \frac{Gm_i}{\rho_i^3}(y_i - y) \\ \ddot{z} &= -z_3\Omega^2 + S + \sum_{i=1}^{3} \frac{Gm_i}{\rho_i^3}(z_i - z) \end{aligned}\right\} \tag{4}$$

where Ω is the angular velocity of the earth-moon mass center around the sun The relationship $R^3\Omega^2 = Gm_3$ has been substituted into Eqs (4) ρ_i is given by the following expression:

$$\rho_i^2 = (x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2 \qquad i = 1,2,3 \tag{5}$$

From Fig 2 it can be seen that the coordinates of the sun (x_3 y_3 z_3) are determined by the expressions

$$\begin{aligned} x_3 &= R(\cos\psi \cos\theta + \cos i \sin\psi \sin\theta) - \xi_p \\ y_3 &= -R(\cos\psi \sin\theta - \cos i \sin\psi \cos\theta) - \eta_p \\ z_3 &= R \sin\psi \sin i \end{aligned} \tag{6}$$

where R is the distance between the sun and the earth-moon mass center, and i is the inclination of the earth-moon orbit plane to the ecliptic The angular positions of the sun and the earth-moon line with respect to the vernal equinox are given, respectively, by ψ and θ The angle θ is measured in the earth-moon orbit plane and ψ is measured in the ecliptic The relationships defining ψ and θ are

$$\begin{aligned} \psi &= \Omega t + \psi_0 \\ \theta &= \omega t + \theta_0 \end{aligned} \tag{7}$$

where ψ_0 and θ_0 are the initial values of ψ and θ

Equations (4–6) were programmed in Fortran 60 compiler language for numerical integration with a standard numerical integration program available from the Computation Center Library at the University of Texas [12] The program uses an Adams-Moulton integration scheme employing partial double precision arithmetic A Runge-

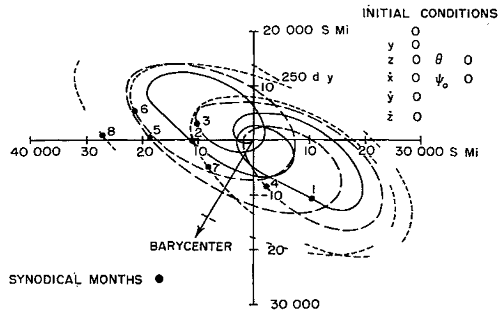

Fig 3 Spacecraft motion about L_4 resulting from zero initial displacement and velocity errors

Kutta integration procedure is used to provide the starting values for the Adams-Moulton procedure The computations were performed on the Control Data Corporation 1604 computer at the University of Texas

The following values were used for the constants in the problem: $m_1 = 4\ 09407 \times 10^{23}$ slugs; $m_2 = 5\ 02154 \times 10^{21}$ slugs; $m_3 = 1\ 36122 \times 10^{29}$ slugs; $G = 2\ 33546 \times 10^{-19}$ mile^3/sec^2-slug; $\Omega = 1\ 99082 \times 10^{-7}$ rad/sec; $R = 9\ 29136 \times 10^7$ mile; $i = 5°9'$; $\psi_0 = 0$; $\theta_0 = 0$; and the earth-moon distance $d = 2\ 38855 \times 10^5$ mile With these values, the calculated value for the angular velocity of the x,y,z coordinate system is $\omega = 2\ 665075637 \times 10^{-6}$ rad/sec

Effects of the Gravitational Attraction of the Sun

It is of interest to determine how the perturbations due to the gravitational attraction of the sun influence the motion of a vehicle placed at one of the libration points with zero relative velocity Unless stated differently, the initial position of the sun will be on the extended earth-moon line with the moon between the earth and sun, i e , $\psi_0 = 0$ and $\theta_0 = 0$ The projection of the vehicle's motion on the x-y plane is shown in Figs 3–5 for a period of 700 days The results were obtained by numerically integrating Eqs (4) with S_x, S_y, and S set equal to zero The dots in the figures represent intervals of one synodical month Note in Fig 3 that after eight months the envelope of motion exceeds 30,000 miles in the x direction and 20,000 miles in the y direction Figures 4 and 5 indicate that the envelope of motion continues to grow Furthermore, it does not appear that after 700 days a limiting value for the envelope is being approached In Fig 5, it can be noted that the envelope of motion appears to be warped around a circle centered at the earth-moon mass center or barycenter It is, also, interesting to note that the irregular initial motion damps out and that there is an approximate one month periodicity associated with the motion shown in Fig 5

Figure 6 illustrates the displacement in the z direction as a function of time for a period of 250 days The amplitude of motion is increasing with time The period of the motion

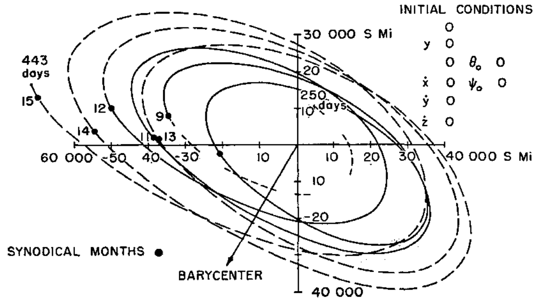

Fig 4 Spacecraft motion about L_4 resulting from zero initial displacement and velocity errors (continued from Fig 3)

in the z direction is about 27 6 days This is near the 29 53 day period for the synodical month

Figure 7 shows the projection of the motion on the x-y plane for the first 250 days following release with zero velocity at L_5 and with ψ_0 and θ_0 both zero It can be seen that after this period the envelope of motion is somewhat larger than the envelope of the motion shown in Fig 3 This is explained by noting that the sun is in the same initial position for both cases and, consequently, makes its nearest approach to L_5 one-half revolution sooner than it does for L_4 However, as in the case of the motion started at L_4, the envelope of motion appears to be growing without bound

Although definite regional instability has not been demonstrated, it is reasonable to expect that unless the trend indicated in these results is reversed, the vehicle will ultimately escape from the L_4- or L_5-centered motion

Effect of the Solar Radiation Pressure

In the study reported here, it is assumed that the solar-radiation pressure is given by

$$\mathbf{S} = -k(A/m)(\boldsymbol{\varrho}_3/\rho_3^3) \tag{8}$$

where $\boldsymbol{\varrho}_3 = \mathbf{r}_3 - \mathbf{r}$, A is the cross-sectional area normal to $\boldsymbol{\varrho}_3$, m is the mass of the vehicle, and k is a constant For this study a value of $k = 4\ 606191 \times 10^{16}$ slug ft/sec^2 was used From Eq (8) it is apparent that the effect of the solar-radiation pressure will depend on the area-to-mass ratio for the vehicle

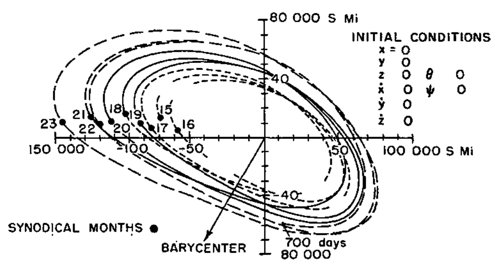

Fig 5 Spacecraft motion about L_4 resulting from zero initial displacement and velocity errors (continued from Fig 4)

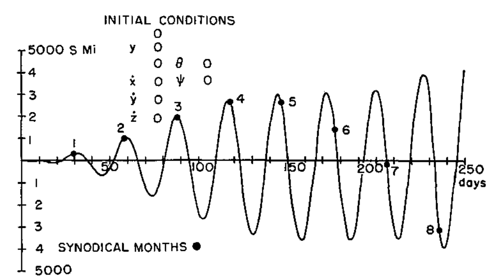

Fig 6 Spacecraft motion in the z direction about L_4 resulting from zero initial displacement and velocity errors

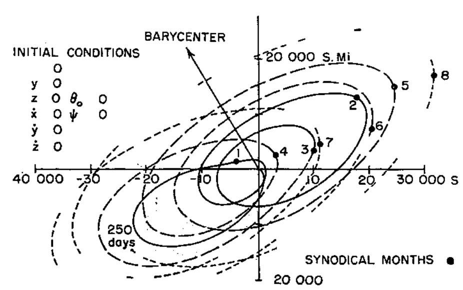

Fig 7 Spacecraft motion about L_5 resulting from zero initial displacement and velocity errors

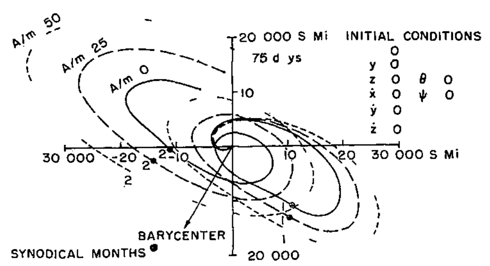

Fig 8 Spacecraft motion about L_4 considering seveıal spacecraft area to mass ratios

Figure 8 shows the projection of the motion on the x-y plane for a vehicle released with zero velocity at L_4 Results for A/m ratios of 0 ft²/slug, 25 ft²/slug, and 50 ft²/slug are shown The results were obtained by numerically integrating Eqs (4) with the appropriate expressions for S_x, S_y, and S_z In general, it can be seen that the solar-radiation pressure causes the vehicle to move farther away from L_4 in a given time The larger A/m ratios aıe associated with large envelopes of motion

Location of Earth-Moon-Sun Equilibrium Points

When the effects of the sun are included, L_4 and L_5 are no longer equilibrium points It is of interest to locate the equilibrium points for various orientations of the earth, moon, and sun The term "equilibrium" is used here to denote a point where a particle at rest with respect to L_4 or L_5 will have zero acceleration in the x,y,z coordinate system The points can be located by setting $\ddot{x} = \ddot{y} = \ddot{z} = \dot{x} = y = j = 0$ in Eqs (4) For a fixed value of time, the position of the sun will be specified by Eqs (6) Hence, the three resulting nonlinear algebraic equations can be solved for the location of the equilibrium points corresponding to the selected configuration

Figure 9 shows the x-y coordinates for a 16 day period The points for A/m ratios of 0 ft²/slug and 100 ft²/slug are shown It is interesting to note that equilibrium points lie along a circle whose center is at the earth-moon mass center This agrees with the general shape of the envelope of motion shown in Fig 5 During the period between 10 and 14 days, the equilibrium point moves more than 140,000 miles away from L_4 in the negative x direction The solar-radiation pressure changes the locations of the equilibrium points, but they still lie very near to the same circle For $A/m = 0$ ft²/slug, the points move along the circle with a period of about 14 7 days, i e , about one-half of the synodical period For $A/m = 100$ ft²/slug, the period required for the equilibrium point to return to its original position is just over 16 days

Effects of the Initial Conditions

Several sets of initial conditions were tried in an effort to find a set which would result in a small envelope of motion

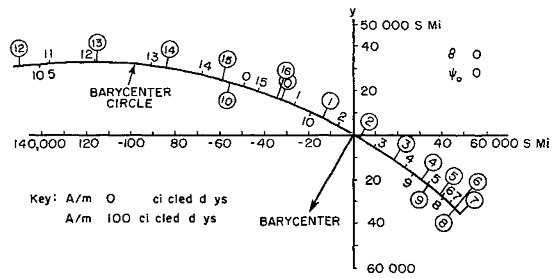

Fig 9 Time dependent equilibrium point location in the L_4 region

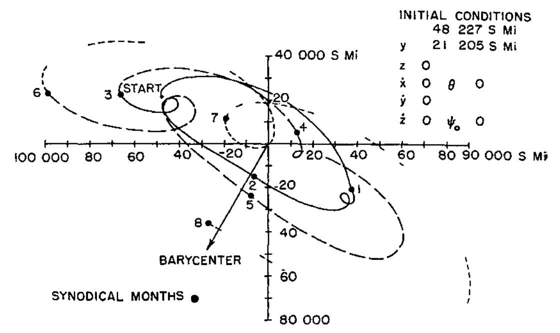

Fig 10 Spacecraft motion about L_4 with the initial displacement corresponding to a time dependent equibrilium point

Figure 10 shows results for $\dot{x} = y = \dot{z} = 0$ with the initial position chosen to coincide with the equilibrium point at $t = 0$, as shown in Fig 9 Figure 11 shows the motion resulting from the conditions $x = y = z = \dot{x} = y = z = 0$ when $t = 2\ 4$ days, i e , $\psi_0 = 2\ 36°$ and $\theta_0 = 31\ 66°$ This corresponds to the time when the equilibrum point is near L_4 In both cases the resulting motion is less stable than the motion shown in Fig 3 This indicates that the amplitude of the resulting motion will depend on the initial position of the sun as well as on the initial position and velocity

Figures 12a, 12b, 12c, and 12d show the motion resulting from an initial displacement of zero and with the respective initial velocities of 1) 10 fps directly away from the barycenter, 2 10 fps normal to the barycenter-L_4 line in the direction of rotation of the coordinate system, 3) 10 fps directly towards the barycenter, and 4) 10 fps in the opposite direction of case 2 It is interesting to note that the envelope of motion for Fig 12c is smaller at any time than the envelope of the motion shown in Fig 3 at a corresponding time

Impulse Required to Induce Stability at L_4

In order to force the particle to stay precisely at L_4, the perturbing force due to the sun must be nulled by application of a continuous thrust The area under the thrust-time curve will represent the magnitude of the specific impulse per unit mass of the vehicle required to force the vehicle to remain at L_4 The value of the impulse required is 2360 lb-sec/slug/yr This is represented by a horizontal line in Fig 13 Figure 13 also shows the impulse requirements necessary to force a vehicle to remain at various points around L_4 The results were obtained with values of $\psi_0 = 0$ and $\theta_0 = 0$ The angular positions of the points are meas ured from the positive x axis and the radial distance is measured from L_4 The L_4 point requires the minimum impulse At points along the barycenter circle no appreciable increase in impulse ıequirement is found Along the L_4-barycenter line, however, extreme increases in specific impulse requirements are necessary as the radial distance is increased or decreased

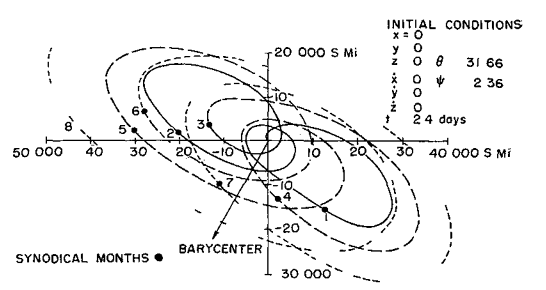

Fig 11 Spacecraft motion about L_4 with zero initial displacement and an initial time

Discussion of Results

Although no definite conclusion concerning the nature of the motion about L_4 can be drawn from the results shown, it is obvious that the effects of the sun cause a vehicle to move through wide departures from L_4 and the application of continuous thrust is required to force a vehicle to remain precisely at L_4 As the earth-moon orbit plane rotates, the L_4 equilibrium point moves along a circle whose center is at the earth-moon mass center At times the displacement of

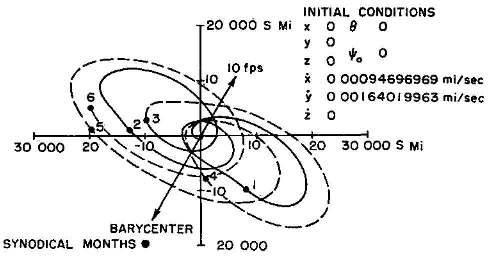

Fig 12a Spacecraft motion about L_4 with an initial velocity of 10 fps at 60°

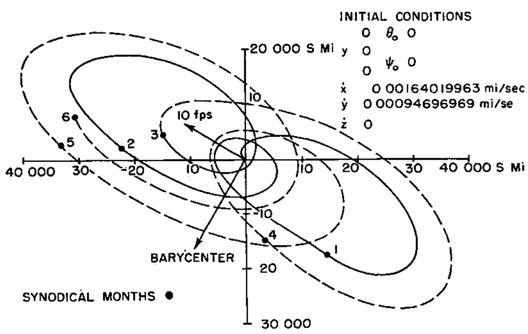

Fig 12b Spacecraft motion about L_4 with an initial velocity of 10 fps at 150°

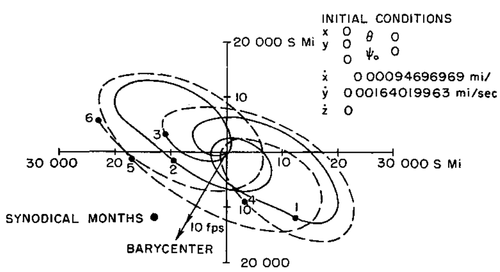

Fig 12c Spacecraft motion about L_4 with an initial velocity of 10 fps at 240°

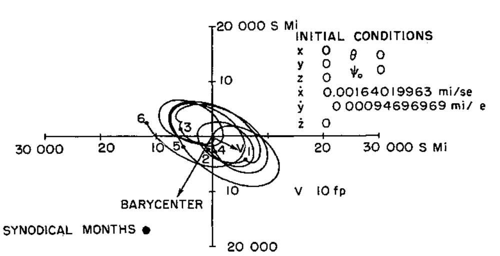

Fig 12d Spaceciaft motion about L_4 with an initial velocity of 10 fps at 330°

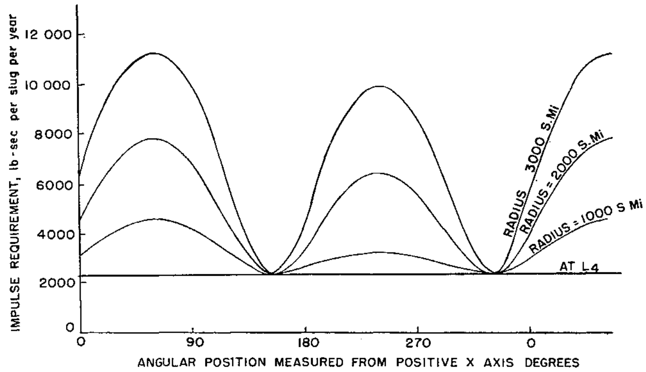

Fig 13 Impulse requirement to induce stability at locations in the L_4 region

the equilibrium point in the negative x direction exceeds 140,000 miles The effects of the initial position of the sun and the values of the initial position and the initial velocity of the vehicle have a decided effect on the motion For the sets of initial conditions considered here, an initial velocity of 10 fps normal to the line joining the L_4 point and the barycenter results in the smallest envelope of motion The impulse requirement for forcing a vehicle to remain precisely at L_4 for a period of one year is found to be 2360 lb-sec/slug The impulse requirements for forcing a vehicle to remain at all other points near L_4 are greater than this value

The results reported here were obtained after neglecting the effects of the eccentricity of the moon's orbit, the nonsphericity of the earth, and the moon and the gravitational attractions of the other planets A more detailed study including these effects should be made A study including the first of these effects would seem to be particularly appropriate Furthermore, since the initial position of the sun is an important parameter in the study, a comparison of the results for different values of this parameter should be made

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